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Application of Schiff's Rotating-Frame Electrodynamics

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The charge distribution and electromagnetic field in a rotating conductor with a net electric charge under stationary conditions are described by Schiff's equations of electrodynamics in a rotating reference frame. The existence of a spatial charge distribution in a conductor at rest in a rotating reference frame is demonstrated.

1. INTRODUCTION

The covariant equations of electrodynamics do not uniquely determine the three-vector formulation of the electromagnetic equations in an arbitrary reference frame. Particularly there exist a large number of different three-vector formulations of electrodynamics in a rotating reference frame (Schiff, 1939; Ise and Uretsky, 1958; Webster, 1963; Irvine, 1964; Modesitt, 1970; Mo, 1970; Bow, 1972, Webster and Whitten, 1973; Landau and Lifshitz, 1975; Corum, 1980). But there are surprisingly few applications of these equations. The main applications have been to the cases of rotating charged shells with cylindrical or spherical symmetry, (Ise and Uretsky, 1958; Corum, 1980) and to different versions of the Sagnac effect (Heer, 1964; Yildiz and Tang, 1966, Post, 1967; Anderson and Ryon, 1969; Volkov and Kiselev, 1970).

In this paper I will consider Schiff's equations (Schiff, 1939). Webster and Whitten (1973) have discussed these equations, and write that their use is often more difficult than describing a rotating object by the ordinary Maxwell's equations, referred to an inertial reference frame. In fact Schiff has told these authors that he regarded as the chief value of his work, the warning it should give anyone to avoid the use of rotating coordinate axes.

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I describe here an isolated, massive, charged, rotating, cylindrical conductor from its rotating rest frame, R, and then transform the description to the inertial rest frame of its axis, I. In this application the use of Schiff's equations is even simpler than describing the conductor directly from I by use of the ordinary Maxwell's equations (Grøn and Vøyenli, 1983).

2. SCHIFF'S EQUATIONS

The covariant formulation of Maxwell's equations is

$$F_{\mu\upsilon,\sigma} + F_{\upsilon\sigma,\mu} + F_{\sigma\mu,\upsilon} = 0 \tag{1}$$

$$\left[(-g)^{1/2} F^{\mu\upsilon}\right]_{,\upsilon} = (-g)^{1/2} J^{\mu} \tag{2}$$

where $F^{\mu\upsilon}$ are the components of the electromagnetic field tensor, J^{μ} of the four-current, and g is the determinant of the metrical tensor.

Let R be rotating relative to the inertial frame J with a constant angular velocity ω. Using Cartesian axes the coordinate transformation from J to R takes the form

$$\begin{aligned} x &= x' \cos \omega t + y' \sin \omega t' \\ y &= -x' \sin \omega t' + y' \cos \omega t' \\ z &= z', \qquad t = t' \end{aligned} \tag{3}$$

Differentiating and substituting the result in the Minkowski line element in J, one finds the line element in R:

$$\begin{aligned} ds^2 = &-\left[1 - \omega^2(x^2 + y^2)\right] dt^2 + 2\omega\, dt\, dx - 2\omega\, dt\, dy \\ &+ dx^2 + dy^2 + dz^2 \end{aligned} \tag{4}$$

(Here and in the following the velocity of light has been put equal to 1.) This gives the following nonvanishing components of the metrical tensor:

$$\begin{aligned} g_{00} &= -1 + \omega^2(x^2 + y^2), \qquad g_{11} = g_{22} = g_{33} = 1 \\ g_{01} &= g_{10} = \omega y, \qquad g_{02} = g_{20} = -\omega x \end{aligned} \tag{5}$$

Since the determinant $g = -1$, the electromagnetic field equations (1) and

(2) are unaltered by the transformation (3). But the connection between the covariant and contravariant components of the field tensor depends on the metrical tensor,

$$F^{\mu\upsilon} = g^{\mu\alpha} g^{\upsilon\beta} F_{\alpha\beta} \tag{6}$$

If we define

$$\begin{aligned} &F_{23} = B_x, \qquad F_{31} = B_y, \qquad F_{12} = B_z \\ &F_{01} = E_x, \qquad F_{02} = E_y, \qquad F_{03} = E_z \end{aligned} \tag{7}$$

it follows from Eq. (1) that Maxwell's source-free equations have the same form in every reference frame,

$$\nabla \cdot \mathbf{B} = 0 \tag{8}$$

$$\nabla \times \mathbf{E} + \partial \mathbf{B}/\partial t = 0 \tag{9}$$

From Eqs. (2), (5), and (7) one finds that in R the source equations (2) have the 3-vector form,

$$\nabla \cdot \mathbf{E} = (\rho + \sigma)/\varepsilon_0 \tag{10}$$

$$\nabla \times \mathbf{B} - \partial \mathbf{E}/\partial t = \mu_0(\mathbf{j} + \mathbf{i}) \tag{11}$$

where ρ and $\mathbf{j}$ are the charge and current densities, respectively, and where

$$\sigma = \nabla \cdot (\mathbf{v} \times \mathbf{B}) = 2\boldsymbol{\omega} \cdot \mathbf{B} - \mathbf{v} \cdot (\nabla \times \mathbf{B}) \tag{12}$$

$$\mathbf{i} = \mathbf{v} x (\nabla \times \mathbf{E}) + \nabla \times [\mathbf{v} \times (\mathbf{E} - \mathbf{v} \times \mathbf{B})] \tag{13}$$

with

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \tag{14}$$

Equations (8)–(14) are Schiff's equations. In the stationary case (9), (10), and (11) reduce to

$$\nabla \times \mathbf{E} = 0 \tag{15}$$

$$\nabla \cdot (\mathbf{E} - \mathbf{v} \times \mathbf{B}) = \rho/\varepsilon_0 \tag{16}$$

$$\nabla \times [\mathbf{B} - \mathbf{v} \times (\mathbf{E} - \mathbf{v} \times \mathbf{B})] = \mu_0 \mathbf{j} \tag{17}$$

If also all charges are at rest in R, Eq. (17) takes the form

$$\nabla \times [\mathbf{B}-\mathbf{v}\times(\mathbf{E}-\mathbf{v}\times\mathbf{B})]=0 \tag{18}$$

In addition to the field equations we shall need the equation of motion in R for a charge q in an electromagnetic field. The covariant form of this equation is

$$\frac{du^\mu}{d\tau}+\Gamma^\mu{}_{\alpha\beta}u^\alpha u^\beta=(q/m)F_\alpha{}^\mu u^\alpha \tag{19}$$

Here u^μ are the components of the four-velocity of the charge, m is its rest mass, $d\tau$ the proper-time interval as measured on a standard clock following the particle, and $\Gamma^\mu{}_{\alpha\beta}$ the Christoffel symbols.

Bow (1972) has deduced the three-vector form of this equation in R for the general case with a moving charge. We shall only need the equation as applied to a charge at rest in R. In this case it takes form

$$\gamma m\boldsymbol{\omega}\times\mathbf{v}=q[\mathbf{E}-(\mathbf{v}\cdot\mathbf{E})\mathbf{v}], \qquad \gamma=(1-v^2)^{-1/2} \tag{20}$$

The solution of the equations (8), (15), (16), (18), and (20) gives the electromagnetic field and charge distribution in R. The transformation to the inertial rest frame of R's axis is given by Modesitt (1970),

$$\rho'=\rho, \qquad \mathbf{j}'=\mathbf{j}+\rho\mathbf{v}$$

$$\mathbf{B}'=\mathbf{B}, \qquad \mathbf{E}'=\mathbf{E}-\mathbf{v}\times\mathbf{B} \tag{21}$$

3. APPLICATION TO A ROTATING CONDUCTOR

An isolated, charged, massive, cylindrical conductor at rest in R is considered. The charge distribution and electromagnetic field of the conductor will be found when the conduction electrons have come to rest relative to the conductor.

From the cylindrical symmetry of the problem it follows that the only nonvanishing components of $\mathbf{E}$ and $\mathbf{B}$ in cylindrical coordinates are

$$E(r)=E_r(r), \qquad B(r)=B_z(r) \tag{22}$$

Thus $\mathbf{v}\perp\mathbf{E}$, so that the last term in equation (20) vanishes, which leads to

$$\gamma m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})=-e\mathbf{E} \tag{23}$$

This immediately gives the electrical field strength in the conductor as measured in R:

$$E = \gamma r \omega B_m \tag{24}$$

where

$$B_m = m\omega/e \tag{25}$$

For $\omega = 10^3\ \mathrm{s}^{-1}$ we get $B_m = 5.7 \times 10^{-10}$ tesla.

Equations (8) and (15) are satisfied by every field of the form given in Eq. (22). Since $\mathbf{v} \cdot \mathbf{B} = 0$, the vector identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \tag{26}$$

gives

$$\mathbf{v} \times (\mathbf{v} \times \mathbf{B}) = -v^2 \mathbf{B} \tag{27}$$

Equation (18) then reduces to

$$\nabla \times \left[(1 - v^2)\mathbf{B} - \mathbf{v} \times \mathbf{E} \right] = 0 \tag{28}$$

which, together with Eq. (22) gives

$$\frac{d}{dr}\left[(1 - v^2) B - r\omega E \right] = 0 \tag{29}$$

Substitution from Eq. (24) and integration gives

$$B = \gamma^2 \left(B_0 - \gamma v^2 B_m \right) \tag{30}$$

where $B_0 = B(0)$.

Applying Stoke's theorem to Eq. (17) gives the generalized form of Ampere's law in integral form, valid for Schiff's formulation of the electromagnetic field equations in a rotating reference frame,

$$\oint [\mathbf{B} - \mathbf{v} \times (\mathbf{E} - \mathbf{v} \times \mathbf{B})] \cdot d\mathbf{r} = \mu_0 J \tag{31}$$

where J is the current enclosed by the integration path.

Equation (31) is now applied to a rectangular path with one side along the axis of the conductor and one parallel to the axis infinitely far from it. The path is midway between the ends of the conductor, and its side along the axis is much shorter than the conductor. Then $\mathbf{v}\times\mathbf{E}$ and $\mathbf{v}\times(\mathbf{v}\times\mathbf{B}) = -v^2\mathbf{B}$ are both directed parallel to the axis at the path, so that the integral along its radial sides vanishes. At infinity $\mathbf{E}=\mathbf{B}=0$. Then Eq. (31) gives

$$B_0 = \mu_0 J = \mu_0 \frac{\lambda\omega}{2\pi} \tag{32}$$

where λ is the net charge per unit length of the conductor.

Substituting Eqs. (24) and (30) into Eq. (16), and using that

$$\nabla\cdot\mathbf{a}(r) = \frac{1}{r}\frac{d}{dr}(ra_r) \tag{33}$$

in cylindrical coordinates, gives

$$\rho = \gamma^4\left[\rho_0 + \varepsilon_0(3\gamma - 2 - \gamma^{-1})\omega B_m\right] \tag{34}$$

where

$$\rho_0 = 2\omega\varepsilon_0(B_m - B_0), \qquad B_0 = \mu_0\lambda\omega/2\pi \tag{35}$$

The electromagnetic field outside the conductor, neglecting end effects, is found from Ampére's law in the form (31) and the generalized Gauss' law

$$\oint\!\!\oint(\mathbf{E} - \mathbf{v}\times\mathbf{B})\cdot d\mathbf{s} = Q/\varepsilon_0 \tag{36}$$

where Q is the net charge enclosed by the integration surface. One finds that there is only a radial electrical field

$$E_{\text{outside}} = \lambda/2\pi\varepsilon_0 r \tag{37}$$

The surface charge per unit length of the conductor, λ_σ, is found from the boundary condition for the electric field on the surface

$$\Delta(\mathbf{E} - \mathbf{v}\times\mathbf{B}) = \sigma/\varepsilon_0 = \lambda_\sigma/2\pi R\varepsilon_0 \tag{38}$$

where R is the radius of the conductor. Substituting from Eqs. (24), (30), (32), and (37) we get

$$\lambda_\sigma = \gamma^2(R)\left[\lambda - v^2(R)\lambda_m\right] \tag{39}$$

where

$$\lambda_m = 2\pi B_m/\mu_0\omega = 2\pi m/\mu_0 e = 2.8\times 10^{-5}\ \text{C/m} \tag{40}$$

is the surface charge per unit·length corresponding to B_m. We see that λ_m may be identified as

$$\lambda_m = e/2r_0 \tag{41}$$

where

$$r_0 = e^2/4\pi\varepsilon_0 m \tag{42}$$

is the classical electron radius. The charge density, current density and electromagnetic field, as measured in the inertial rest frame, J, of the axis of the conductor, is found from the transformation equations (21). It follows that the charge density and the magnetic field are given by Eqs. (16) and (14), respectively. The current density is

$$\mathbf{j}' = \rho\mathbf{v} \tag{43}$$

while the electrical field strength is

$$E' = \gamma^2 r\omega(\gamma B_m - B_0) \tag{44}$$

From Eqs. (25) and (32) is seen that B_m is connected with the inertia of the conduction electrons, and B_0 with the net charge of the conductor. Equations (30) and (44) show that the electromagnetic field in the conductor has one component due to its net charge and one due to the inertia of the electrons.

To estimate the magnitude of these two components we note that under normal laboratory conditions air ionizes in an electrical field of the order 10^6 V/m. From Eq. (37) we find that for a cylinder with a radius $R = 10^{-1}$ m this corresponds to a value of λ of the order 10^{-5} C/m. This is the maximum charge per unit length the cylinder will keep without leakage. If we choose $\omega = 10^3$ s^{-1} we find a corresponding value of B_0 of the order 10^{-9} tesla, which is of the order of B_m calculated earlier under the same conditions. Hence we see that the contribution to **E** and **B** in the conductor, from its largest possible net charge and from the inertia of the conduction electrons, are of the same order under laboratory conditions. The potential difference between the axis and the surface of the cylinder is given by

$$U = \int_0^R E' dr \tag{45}$$

For $R\omega \ll 1$ we find

$$U \cong (\mu_0/4\pi)(\lambda - \lambda_m)R^2\omega^2 \tag{46}$$

With $\lambda = 10^{-5}$ C/m, $R = 10^{-1}$, and $\omega = 10^3$ s^{-1}, this gives $U = 1.8 \times 10^{-8}$ V.

4. CONCLUSION

The electromagnetic field and charge distribution in a charged, rotating, cylindrical conductor have been calculated in the rotating rest frame of the conductor, by means of Schiff's generalization of Maxwell's equations. Compared to the ordinary Maxwell equations, valid in inertial frames, Schiff's source equations have additional terms. Applied to a charged conductor at rest in a rotating reference frame these terms imply that a nonvanishing spatial charge distribution will exist within the conductor. As described in the inertial rest frame of the axis of the conductor, the spatial charge distribution in the conductor is a result of the centrifugal effect on the conduction electrons and the magnetic field caused by the motion of the charge (Grøn and Vøyenli, 1983).

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